NOTIZEN

An Inertial Contribution to the Steady State Film Profiles of Viscous Drainage

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The effect of inertial terms in the Navier-Stokes equation of viscous drainage on a flat plate is investigated. It is found that they give rise to a constant correction term to the Jeffreys' parabola which persists in the steady state.

By virtue of its wide-ranging applications in physics, chemistry and technology, the problem of viscous lifting and drainage has been the object of considerable theoretical and experimental study [1–6]. In its simplest form, it consists in determining the profile of the thin film adhering to the surface of a flat plate immersed in a bath of wetting liquid which is allowed to drain freely under the action of gravitational and viscous forces.

Neglecting the effects of surface tension, this can be treated as a flow problem in which the velocity is in the y-direction and a function merely of x and t (cf. Fig. 1). The Navier-Stokes equation, then, takes the form

$$\partial v/\partial t = \nu\, \partial^2 v/\partial x^2 - g\,, \qquad (1)$$

where ν is the kinematic viscosity and g the acceleration due to gravity. Together with the initial condition

$$v(x, 0) = 0\,, \qquad (2\,\text{a})$$

the velocity is required to satisfy the boundary conditions

$$v(0, t) = 0 \quad \text{and} \quad \frac{\partial v}{\partial x}(h, t) = 0\,, \qquad (2\,\text{b, c})$$

stipulating that there be no slipping of the liquid on the surface of the plate, and no shear force on the free film surface $x = h(y, t)$.

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Equation (1) can be solved by a separation of variables, and the solution takes the form

$$v(x, t) = \frac{g}{2\nu}(x^2 - 2xh) + \frac{2gh^2}{\nu} \cdot \sum_{n=0}^{\infty} [\exp\{-\alpha_n^2 \nu t/h^2\}/\alpha_n^3] \sin \frac{\alpha_n x}{h}\,, \qquad (3)$$

where $_n = (n + \tfrac{1}{2})\pi$. With the velocity given by Eq. (3) the rate of flow of the liquid can be obtained as

$$q = \int_0^h v\, \mathrm{d}x = -\frac{gh^3}{3\nu} + \frac{2gh^3}{\nu} \cdot \sum_{n=0}^{\infty} [\exp\{-\alpha_n^2 \nu t/h^2\}/\alpha_n^4]\,. \qquad (4)$$

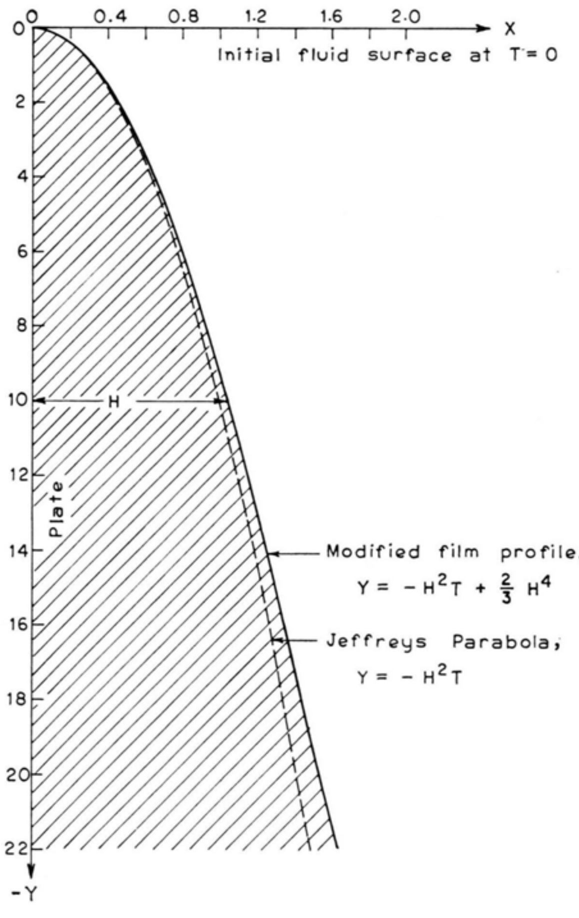

Fig. 1. Steady state film profiles at $T = 10$.

The thickness of the liquid film is related to the flow rate through the equation of continuity which, in this case, takes the form

$$\frac{\partial q}{\partial y}+\frac{\partial h}{\partial t}=0\,. \qquad (5)$$

Since Eq. (4) gives q as a function of h, Eq. (5) can be rewritten as

$$\left(\frac{\partial q}{\partial h}\right)\left(\frac{\partial h}{\partial y}\right)+\frac{\partial h}{\partial t}=0\,. \qquad (6)$$

This partial differential equation is of the Lagrange form with the solution

$$y=\int\left(\frac{\partial q}{\partial h}\right)_t \mathrm{d}t+\Phi(h)\,, \qquad (7)$$

giving the film thickness h as an implicit function of y and t. $\Phi(h)$ is determined by the requirement that at $t=0$, Eq. (7) should describe the initial shape of the liquid surface $(y=0)$. In terms of the non-dimensional quantities

$$H=h(g/\nu^2)^{1/3}\,,\quad Y=y(g/\nu^2)^{1/3}\quad \text{and}$$
$$T=t(g^2/\nu)^{1/3}\,, \qquad (8)$$

we thus obtain the required relation

$$Y=-H^2T+\tfrac{2}{3}H^4-2H^2\sum_{n=0}^{\infty}[\exp\{-a_n^2T/H^2\}/a_n^6]\cdot(5H^2+2a_n^2T)\,, \qquad (9)$$

depicting the time development of the liquid film profile.

If one were to start without the inertial term $\partial v/\partial t$ in Eq. (1) and carry through the derivation, one would obtain the well-known Jeffreys' solution

$$Y=-H^2T\,, \qquad (10)$$

according to which the liquid film in the steady state has a parabolic profile. In as much as the effect of the inertial term is only to introduce exponentially decaying transient terms, one might conclude that it should not alter the steady state profiles. However, an inspection of Eq. (9) reveals that it does not reduce to Eq. (10) as $t\to\infty$ due to the presence of the middle term $\frac{2}{3}H^4$. Going back a few steps, we see that the solution (3) of the Navier-Stokes equation itself does go over to its steady state value as $t\to\infty$. This, however, is all that can be legitimately demanded. On the other hand, the calculation of the film thickness is twice removed from the basic Eq. (1), in that its solution is subjected to a further integration (7) over time, via the equation of continuity. As a result, although the flow rate settles down to a steady value as soon as the transients die down, these transients, while they were active, contribute a net correction term to the steady state film thickness, which is an integrated effect of the total flow rate. That the correction $\frac{2}{3}H^4$ owes its origin to a cumulative effect of the inertial term is also borne out by the fact that it is more prominent at larger values of density and smaller values of viscosity. The presence of this term distorts the parabolic profiles at large values of H (Figure 1).

Numerous modifications of this basic problem of viscous drainage are of wide applicability, e. g. in lubrication, electroplating, dip-coating etc. [7]. Hydromagnetic versions of some of these problems have also appeared in the literature [8, 9]. In all these cases, a consideration of the inertial effects would bring in similar correction terms to the steady state profiles.

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